GOSFORD HIGH SCHOOL

2016
TRIAL HSC EXAMINATION.

MATHEMATICS

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Board approved calculators may be used
- Write using black pen
- A reference sheet is provided
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I – 10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section

Section I

10 marks

Attempt Questions 1 – 10

Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1 -10.

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1 What is 52.09684 correct to 4 significant figures?

(A) 52.0968

(B) 52.09

(C) 52.10

(D) 52.1

2 Which of the following is equal to $\frac{\sqrt{3}}{2\sqrt{3}+\sqrt{2}}$?

(A) $\frac{6-\sqrt{6}}{4}$

(B) $\frac{6-\sqrt{6}}{10}$

(C) $\frac{6+\sqrt{6}}{10}$

(D) $\frac{3-\sqrt{6}}{5}$

3 The quadratic equation $x^2 - 3x + 1 = 0$ has roots α and β. What is the value of $\alpha^2 + \beta^2$?

(A) 11

(B) 7

(C) 9

(D) -11

4 A geometric series has $T_1 = \log 3$ and $T_2 = \log 9$. If $T_3 = \log x$, what is the value of x?

(A) 27

(B) 12

(C) 15

(D) 81

5 Let $a = e^x$. Which expression is equal to $log_e(a^2)$?

(A) x^2

(B) e^{x^2}

(C) e^{2^x}

(D) $2x$

6 What are the amplitude and period of the function $f(x) = 2 - sin2x$?

(A) Amplitude 1, period π

(B) Amplitude 1, period 2π

(C) Amplitude 2, period π

(D) Amplitude 2, period 2π

7 What is the value of

$$\sum_{k=1}^{4}(-1)^k k^2$$

(A) -30

(B) -10

(C) 10

(D) 30

8 Which of the following trigonometric expressions is equivalent to $tan\left(\frac{\pi}{2} - x\right)$?

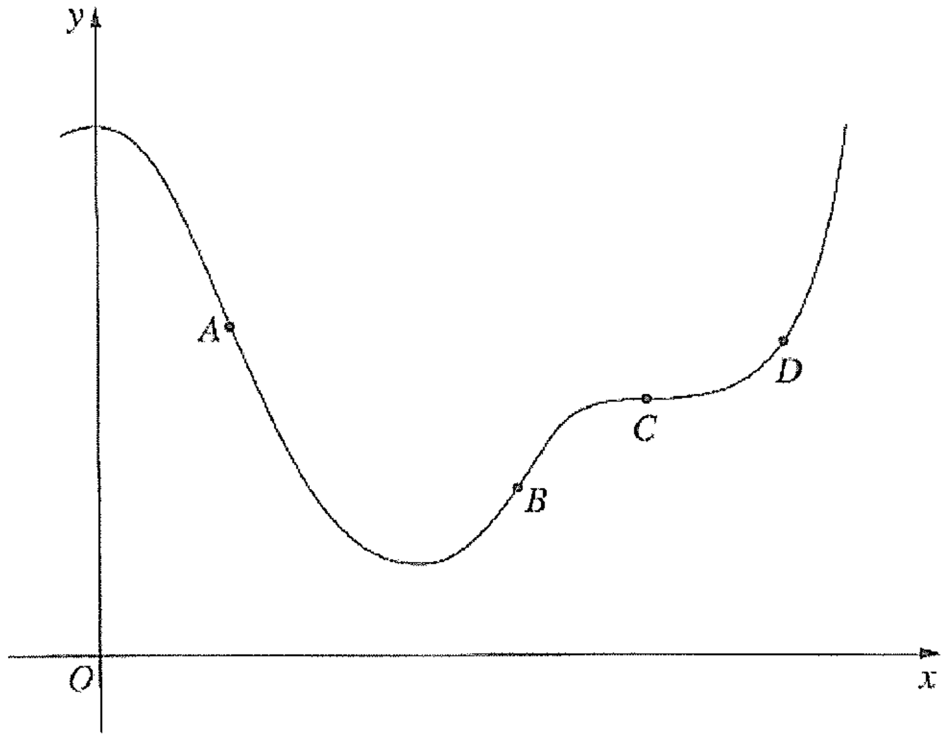
(A) $tanx$

(B) $cotx$

(C) $-tanx$

(D) $-cotx$

9 The diagram shows the points A, B, C and D on the graph $y = f(x)$.



At which point is $f'(x) > 0$ and $f''(x) = 0$?

(A) A

(B) B

(C) C

(D) D

10 A particle is moving along the x axis. The displacement of the particle at time t seconds is x metres.

At a certain time, $\dot{x} = -3\ ms^{-1}$ and $\ddot{x} = -2\ ms^{-2}$.

Which statement describes the motion of the particle at that time?

(A) The particle is moving to the right with increasing speed.

(B) The particle is moving to the left with increasing speed.

(C) The particle is moving to the right with decreasing speed.

(D) The particle is moving to the left with decreasing speed.

Section II

90 marks

Attempt Questions 11 – 16

Allow about 2 hours and 45 minutes for this section

Answer each question in a new writing booklet.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a new booklet.

(a) Factorise $40x^3 - 5$ **2**

(b) Solve $|2x - 1| \leq 3$ **2**

(c) Simplify **2**

$$\frac{9^n \times 15^{2-2n}}{25^{1-n}}$$

(d) Differentiate with respect to x:

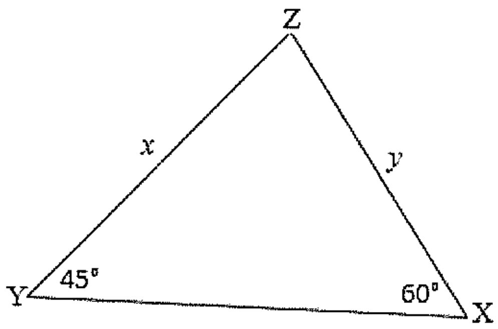
(i) $ln(2x + 1)$ **1**

(ii) $(e^{3x} + 2)^4$ **2**

(iii) $\frac{sinx}{x^3}$ **2**

Question 11 continues on next page

(e)



In the diagram, XYZ is a triangle where $\angle ZYX = 45°$ and $\angle ZXY = 60°$.

Find the exact value of the ratio $\frac{x}{y}$. **2**

(f) Solve $log_2(3x - 4) = 5$ **2**

Question 12 starts on next page

Question 12 (15 marks) Start a new booklet.

(a) Find the coordinates of the focus of the parabola $x^2 = 12(y-3)$ **2**

(b) Find $\int \frac{6x}{x^2-3}\,dx$ **2**

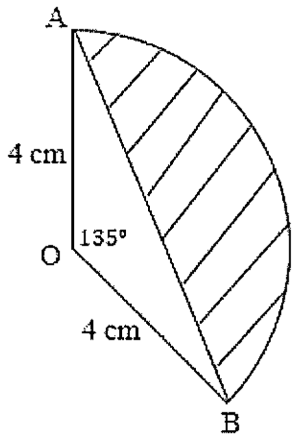
(c) Evaluate

$$\lim_{x\to 3} \frac{x^2-2x-3}{x-3}$$

2

(d) The diagram shows a sector of a circle, centre O. *AB* is a chord of the circle.

If ∠AOB =135°, show that the area of the shaded region is $(6\pi - 4\sqrt{2})cm^2$. **2**



(e) If $f'(x) = 2x + 7$ and $y = f(x)$ passes through the point $(1,4)$, find $f(x)$. **2**

(f) Find the equation of the tangent to the curve $y = 3x^2 + 2x + 1$ at the point $(1,6)$.

3

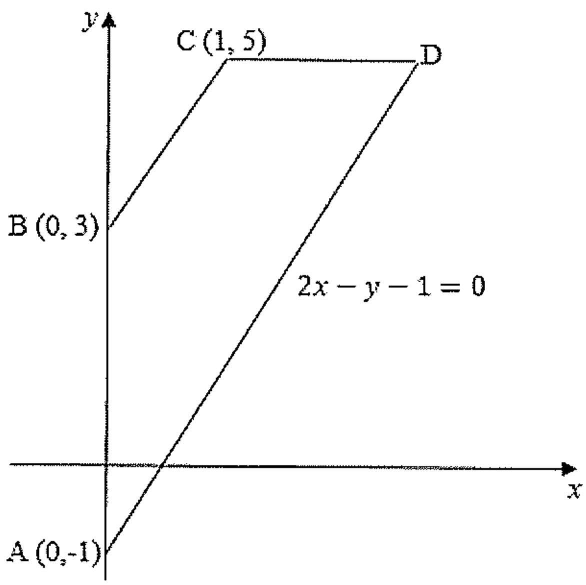
(g) Solve the following for *x*:

$$e^{2x} + 3e^x - 10 = 0$$ **2**

Question 13 starts on next page

Question 13 (15 marks) Start a new booklet.

(a)



In the diagram, *ABCD* is a quadrilateral. The equation of the line *AD* is $2x - y - 1 = 0$.

(i) Show that *ABCD* is a trapezium by showing that *BC* is parallel to *AD*. **2**

(ii) The line *CD* is parallel to the x axis. Find the coordinates of *D*. **1**

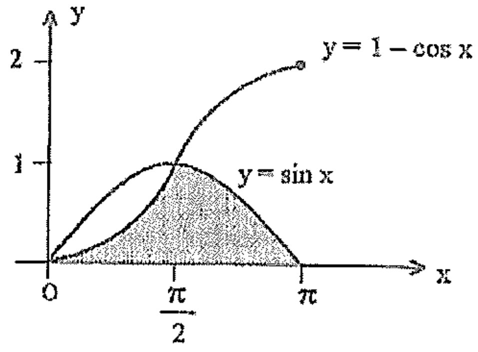
(iii) Find the length of *BC*. **1**

(iv) Show that the perpendicular distance from *B* to *AD* is $\frac{4}{\sqrt{5}}$. **2**

(v) Hence, or otherwise, find the area of the trapezium *ABCD*. **2**

Question 13 continues on next page

(b)



The diagram above shows the graphs of the functions $y = 1 - cosx$ and $y = sinx$, between $x = 0$ and $x = \pi$. The two graphs intersect at the point where $x = \frac{\pi}{2}$.
Evaluate the area of the shaded region. **4**

(c) If $y = ln\left[\frac{1-x}{1+x}\right]$, show that $\frac{dy}{dx} = \frac{-2}{1-x^2}$.

Hence or otherwise, evaluate

$$\int_0^{\frac{1}{2}} \frac{dx}{1 - x^2}$$

3

Question 14 starts on next page

Question 14 (15 marks) Start a new booklet.

(a) (i) Show that $cos\theta tan\theta = sin\theta$ **1**

(ii) Hence solve $8sin\theta cos\theta tan\theta = cosec\theta$ for $0 \leq \theta \leq 2\pi$ **2**

(b) A function is given by $f(x) = 3x^2 - x^3 + 9x - 2$.

(i) Find the coordinates of any stationary points and determine their nature. **3**

(ii) Find the coordinates of any points of inflexion. **2**

(iii) Sketch the graph of $y = f(x)$ for $-2 \leq x \leq 5$. **2**

(iv) For what values of x over the given domain is the function concave up? **1**

(c) Find the volume of the solid formed when the area between the curve $y = ln(2x)$ and the y axis is rotated about the y axis from $y = 1$ to $y = 6$. **4**

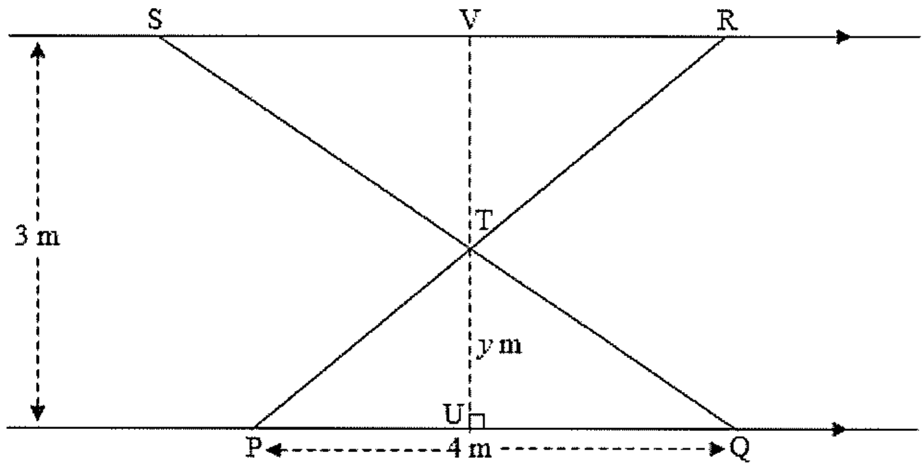
Question 15 starts on next page

Question 15 (15 marks) Start a new booklet.

(a) State the domain and range of the function $y = \sqrt{25 - x^2}$. **2**

(b) Find the value(s) of k for which $x^2 - (k-2)x + 3(k-2) = 0$ has no real roots. **3**

(c)



In the diagram, PQ and SR are parallel railings which are 3 metres apart. The points P and Q are fixed 4 metres apart on the lower railing. Two crossbars PR and QS intersect at T as shown in the diagram. The line through T perpendicular to PQ intersects PQ at U and SR at V. The length of UT is y metres.

(i) By using similar triangles, or otherwise, show that $\frac{SR}{PQ} = \frac{VT}{UT}$. **4**

(ii) Show that $SR = \frac{12}{y} - 4$. **1**

(iii) Hence, show that the total area, A, of ΔPTQ and ΔRTS is

$$A = 4y - 12 + \frac{18}{y}$$ **2**

(iv) Find the value of y that minimises A. **3**

Question 16 starts on next page

Question 16 (15 marks) Start a new booklet.

(a) (i) Use Simpson's rule with 3 function values to find an approximation to the area under the curve $y = \frac{1}{x}$ between $x = a$ and $x = 3a$, where a is positive. **2**

(ii) Using the result in part (i), show that $ln3 \doteqdot \frac{10}{9}$ **1**

(b) A particle is initially at rest at the origin. Its acceleration as a function of time, t, is given by $\ddot{x} = 4sin2t$.

(i) Show that the velocity of the particle is given by $\dot{x} = 2 - 2cos2t$. **2**

(ii) Sketch the graph of the velocity $0 \leq t \leq 2\pi$ and determine the time at which the particle first comes to rest after $t = 0$. **3**

(iii) Find the distance travelled by the particle in the first $\frac{2\pi}{3}$ seconds. **2**

(c) On the 1st January 2000, Toby deposited \$15 000 into a bank account that paid interest at a fixed rate of 4% per annum compounded annually. He later decided to add \$5000 to his account on 1st January each year, starting on 1st January 2007.

(i) Write an expression for the amount in the account on 1st January 2007 after the payment of interest and the first \$5000 deposit. **1**

(ii) How much was in Toby's account on 1st January 2016 after the payment of interest and the \$5000 deposit? **4**

End of examination

Section I

Q1 (C)

2. $\frac{\sqrt{3}}{2\sqrt{3}+\sqrt{2}} \times \frac{2\sqrt{3}-\sqrt{2}}{2\sqrt{3}-\sqrt{2}}$

$= \frac{6-\sqrt{6}}{12-2}$

$= \frac{6-\sqrt{6}}{10}$ (B)

3. $\alpha+\beta = 3$, $\alpha\beta = 1$

$\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= 9-2$

$= 7$ (B)

4. $T_1 = \log 3$

$T_2 = \log 3^2$

$= 2\log 3$. $\therefore r = 2$

$\log x = 4\log 3$.

$= \log 3^4$

$\therefore x = 81$ (D)

5. $\log_e(a^2) = 2\log_e a$

$= 2\log_e e^x$.

$= 2x$ (D)

6. (A)

7. $\sum_{k=1}^{4} (-1)^k k^2 = -1+4-9+16$

$= 10$ (C)

8. (B)

9. (B)

10. (B)

Section II

Question 11

a) $40x^3 - 5 = 5(8x^3-1)$

$= 5(2x-1)(4x^2+2x+1)$

b) $|2x-1| \le 3$

$2x-1 \le 3$ or $2x-1 \ge -3$.

$2x \le 4$ $\quad$ $2x \ge -2$

$x \le 2$ $\quad$ $x \ge -1$

$-1 \le x \le 2$

c) $\frac{9^n \times 15^{2-2n}}{25^{1-n}}$

$= \frac{3^{2n} \times 3^{2-2n} \times 5^{2-2n}}{5^{2-2n}}$

$= 3^2$

$= 9$

d) i) $\frac{d}{dx}\ln(2x+1) = \frac{2}{2x+1}$

ii) $\frac{d}{dx}(e^{3x}+2)^4 = 4(e^{3x}+2)^3 . 3e^{3x}$

$= 12e^{3x}(e^{3x}+2)^3$

iii) $\frac{d}{dx}\frac{\sin x}{x^3} = \frac{x^3 . \cos x - 3x^2 \sin x}{x^6}$

$= \frac{x^2(x\cos x - 3\sin x)}{x^6}$

$= \frac{x\cos x - 3\sin x}{x^4}$

e) $\frac{x}{\sin 60°} = \frac{y}{\sin 45°}$

$\frac{x}{\sqrt{3}/2} = \frac{y}{1/\sqrt{2}}$

$\frac{2x}{\sqrt{3}} = \sqrt{2}y$

$\frac{x}{y} = \frac{\sqrt{6}}{2}$

f) $\log_2(3x-4) = 5$

$3x-4 = 2^5$

$3x-4 = 32$

$3x = 36$

$x = 12$

Question 12

a) $a = 3$. Vertex $(0,3)$

$\therefore$ Focus $(0,6)$

b) $\int \frac{6x}{x^2-3}\,dx$

$= 3\int \frac{2x}{x^2-3}\,dx$.

$= 3\ln(x^2-3) + C$

c) $\lim_{x\to 3} \frac{x^2-2x-3}{x-3}$

$= \lim_{x\to 3} \frac{(x-3)(x+1)}{x-3}$

$= \lim_{x\to 3} x+1$

$= 4$

d)

$A = \frac{1}{2} r^2 (\theta - \sin\theta)$

$= \frac{1}{2} \times 16 \left(\frac{3\pi}{4} - \sin\frac{3\pi}{4}\right)$

$= 8\left(\frac{3\pi}{4} - \frac{1}{\sqrt{2}}\right)$

$= 6\pi - \frac{8}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= (6\pi - 4\sqrt{2})\text{ cm}^2$

e) $f'(x) = 2x + 7$

$f(x) = \int 2x + 7\,dx$

$= x^2 + 7x + C$

sub $(1, 4)$

$4 = 1 + 7 + C$

$C = -4$

$\therefore f(x) = x^2 + 7x - 4$

f) $y = 3x^2 + 2x + 1$

$y' = 6x + 2$

at $x = 1$, $y' = 6 + 2$
$= 8$

$\therefore y - 6 = 8(x - 1)$

$y - 6 = 8x - 8$

$y = 8x - 2$

g) $e^{2x} + 3e^x - 10 = 0$

let $m = e^x$

$m^2 + 3m - 10 = 0$

$(m + 5)(m - 2) = 0$

$m + 5 = 0$ or $m - 2 = 0$

$m = -5$ $\quad m = 2$

$\therefore e^x = -5$ or $e^x = 2$

no solns. $\quad \ln e^x = \ln 2$

$x = \ln 2$

Question 13

a) i) $m_{BC} = \frac{5-3}{1-0}$

$= 2$

$2x - y - 1 = 0$

$y = 2x - 1$ $\quad \therefore m_{AD} = 2$

$\therefore BC \parallel AD$

$\therefore ABCD$ is a trapezium

(one pair of opposite sides parallel)

ii) sub $y = 5$ into $2x - y - 1 = 0$

$2x - 5 - 1 = 0$

$2x = 6$

$x = 3.$

$\therefore D(3, 5)$

iii) $BC = \sqrt{(1-0)^2 + (5-3)^2}$

$= \sqrt{1 + 4}$

$= \sqrt{5}$

iv) $d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$

$= \frac{|0 + -3 - 1|}{\sqrt{4 + 1}}$

$= \frac{4}{\sqrt{5}}$

v) $d_{AD} = \sqrt{(3-0)^2 + (5+1)^2}$

$= \sqrt{9 + 36}$

$= \sqrt{45}$

$= 3\sqrt{5}$

$A = \frac{4}{2\sqrt{5}}(3\sqrt{5} + \sqrt{5})$

$= 8$ sq units.

b) $A = \int_0^{\pi/2} 1 - \cos x\,dx + \int_{\pi/2}^{\pi} \sin x\,dx$

$= \left[x - \sin x\right]_0^{\pi/2} + \left[-\cos x\right]_{\pi/2}^{\pi}$

$= \left(\frac{\pi}{2} - \sin\frac{\pi}{2}\right) - (0 - \sin 0) + (-\cos\pi) - \left(-\cos\frac{\pi}{2}\right)$

$= \frac{\pi}{2} - 1 + 1$

$= \frac{\pi}{2}$ units^2.

c) $y = \ln\left[\frac{1-x}{1+x}\right]$

$y = \ln(1-x) - \ln(1+x)$

$\frac{dy}{dx} = \frac{-1}{1-x} - \frac{1}{1+x}$

$= \frac{-(1+x) - (1-x)}{(1-x)(1+x)}$

$= \frac{-2}{1-x^2}$

$\int_0^{1/2} \frac{dx}{1-x^2}$

$= -\frac{1}{2}\int_0^{1/2} \frac{-2}{1-x^2}\,dx$

$= -\frac{1}{2}\left[\ln\left[\frac{1-x}{1+x}\right]\right]_0^{1/2}$

$= -\frac{1}{2}\left[\ln\frac{1/2}{3/2} - \ln\frac{1}{1}\right]$

$= -\frac{1}{2}\ln\frac{1}{3}$

$= -\frac{1}{2}(\ln 1 - \ln 3)$

$= \frac{\ln 3}{2}$

Question 14

a) i) $\cos\theta\tan\theta = \cos\theta\frac{\sin\theta}{\cos\theta}$

$= \sin\theta$

ii) $8\sin\theta\cos\theta\tan\theta = \operatorname{cosec}\theta$

$8\sin^2\theta = \frac{1}{\sin\theta}$

$8\sin^3\theta = 1$

$\sin^3\theta = \frac{1}{8}$

$\sin\theta = \frac{1}{2}$, $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

b) $f(x) = 3x^2 - x^3 + 9x - 2$

i) $f'(x) = 6x - 3x^2 + 9$

$f''(x) = 6 - 6x$

For stationary points, $f'(x) = 0$

$3x^2 - 6x - 9 = 0$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = 3, -1$

when $x = 3$,

$f(x) = 27 - 27 + 27 - 2$

$= 25$

and $f''(x) = 6 - 18$

< 0

$\therefore$ max at $(3, 25)$

when $x = -1$

$f(x) = 3 + 1 - 9 - 2$

$= -7$

and $f''(x) = 6 + 6$

> 0

$\therefore$ min at $(-1, -7)$

ii) Possible point of inflexion when $f''(x) = 0$

$6 - 6x = 0$

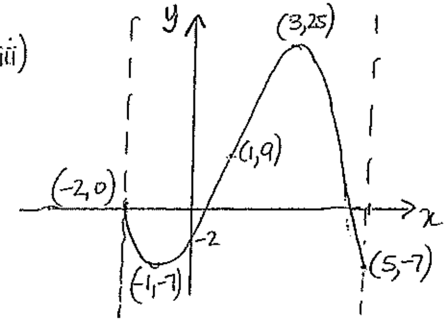
$x = 1$

test change in concavity

x	1^-	1	1^+
$f''(x)$	+	0	−

$\therefore$ point of inflexion at $(1, 9)$

iii)



iv) concave up for $-2 \leq x < 1$

c) $y = \ln(2x)$

$2x = e^y$

$x = \frac{e^y}{2}$

$V = \pi \int_1^6 x^2 \, dy$

$= \pi \int_1^6 \frac{e^{2y}}{4} \, dy$

$= \frac{\pi}{4} \left[\frac{e^{2y}}{2} \right]_1^6$

$= \frac{\pi}{8} \left[e^{12} - e^2 \right]$ units^3

Question 15

a) $f(x) = \sqrt{25 - x^2}$

$25 - x^2 \geq 0$

$x^2 \leq 25$

$-5 \leq x \leq 5$ Domain

Range $0 \leq y \leq 5$

b) $x^2 - (k-2)x + 3(k-2) = 0$

no real roots $\rightarrow \Delta < 0$

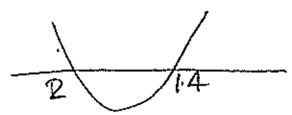
$b^2 - 4ac < 0$

$(k-2)^2 - 4 \cdot 3(k-2) < 0$

$k^2 - 4k + 4 - 12k + 24 < 0$

$k^2 - 16k + 28 < 0$

$(k-14)(k-2) < 0$



$\therefore 2 < k < 14$.

c)

i) In ΔPQT and ΔRST

$\angle SRT = \angle QPT$ (alternate angles $SR \parallel PQ$)

$\angle STR = \angle QTP$ (vertically opposite angles equal)

$\therefore \frac{SR}{PQ} = \frac{RT}{PT}$ corresponding sides in similar Δs

and $\frac{RT}{PT} = \frac{VT}{UT}$ ratio of intercepts

$\therefore \frac{SR}{PQ} = \frac{VT}{UT}$.

(ii) $\frac{SR}{4} = \frac{3-y}{y}$.

$SR = \frac{12 - 4y}{y}$.

$= \frac{12}{y} - 4$

iii) Area $\Delta PTQ = \frac{1}{2} \times 4 \times y$

$= 2y$

Area $\Delta RTS = \frac{1}{2}(3-y)\left(\frac{12}{y} - 4\right)$

$= \frac{1}{2}\left(\frac{36}{y} - 12 - 12 + 4y\right)$

$= \frac{18}{y} - 12 + 2y$

$\therefore$ total area $= 4y - 12 + \frac{18}{y}$

iv) $A = 4y - 12 + 18y^{-1}$

$\frac{dA}{dy} = 4 - 18y^{-2}$ $\quad \frac{d^2A}{dy^2} = 36y^{-3}$

$\frac{dA}{dy} = 0$

$4 - \frac{18}{y^2} = 0$

$\frac{18}{y^2} = 4$.

$y^2 = \frac{18}{4}$

$= \frac{9}{2}$

$y = \pm \frac{3}{\sqrt{2}}$ disregard −ve as y is length.

when $y = \frac{\sqrt{3}}{2}$, $\frac{d^2A}{dy^2} > 0$

$\therefore$ minimum occurs when $y = \frac{\sqrt{3}}{2}$

Question 16

a) i)

x	a	$2a$	$3a$
$\frac{1}{x}$	$\frac{1}{a}$	$\frac{1}{2a}$	$\frac{1}{3a}$

$$\int_a^{3a} \frac{1}{x}\,dx \doteq \frac{a}{3}\left[\frac{1}{a}+4\cdot\frac{1}{2a}+\frac{1}{3a}\right]$$

$$= \frac{a}{3}\left[\frac{10}{3a}\right]$$

$$= \frac{10}{9}$$

ii) $\int_a^{3a} \frac{1}{x}\,dx = \left[\ln x\right]_a^{3a}$

$= \ln 3a - \ln a$

$= \ln \frac{3a}{a}$

$= \ln 3$. $\quad \therefore \ln 3 \doteq \frac{10}{9}$

b) $\dot{x} = 4\sin 2t$

i). $\dot{x} = \int 4\sin 2t \; dt$

$= -2\cos 2t + C$

when $t=0$, $\dot{x}=0$

$0 = -2\cos 0 + C$

$C = 2$

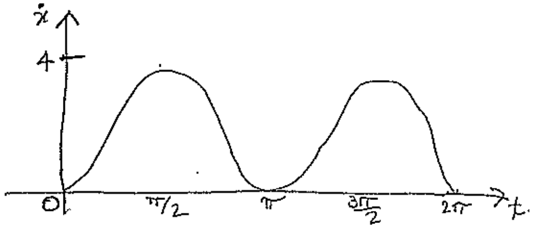
$\therefore \dot{x} = 2 - 2\cos 2t$

ii) Period $= \frac{2\pi}{2}$
$= \pi$.

Range: $-1 \le -\cos t \le 1$

$-2 \le -2\cos t \le 2$

$0 \le 2 - 2\cos t \le 4$



at rest when $\dot{x}=0$

ie $t=\pi$

iii).

$$d = \int_0^{\frac{2\pi}{3}} (2 - 2\cos 2t)\,dt$$

$$\doteq \left[2t - \sin 2t\right]_0^{\frac{2\pi}{3}}$$

$$= \left(\frac{4\pi}{3} - \sin\frac{4\pi}{3}\right) - (0 - \sin 0)$$

$$= \frac{4\pi}{3} + \frac{\sqrt{3}}{2}$$

c) i). $A = 15000 \times 1.04^7 + 5000$

ii) $A_2 = (15000 \times 1.04^7 + 5000) \times 1.04 + 5000$

$= 15000 \times 1.04^8 + 5000 \times 1.04 + 5000$

$A_3 = (15000 \times 1.04^8 + 5000 \times 1.04 + 5000) \times 1.04 + 5000$

$= 15000 \times 1.04^9 + 5000 \times 1.04^2 + 5000 \times 1.04 + 5000$

$\vdots$

$A_{10} = 15000 \times 1.04^{16} + 5000 \times 1.04^9 + 5000 \times 1.04^8 + \dots + 5000$

$= 15000 \times 1.04^{16} + 5000(1 + 1.04 + 1.04^2 + \dots + 1.04^9)$

$= 15000 \times 1.04^{16} + 5000\left[\frac{1(1.04^{10} - 1)}{1.04 - 1}\right]$

$= 28094.71869 + 5000[12.00610712]$

$= 88125.2543$

$= \$88125.25$